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# STRESS INTENSITY FACTORS OF A RADIAL CRACK IN A COMPOUND DISK SUBJECTED TO POINT LOADS

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**Abstract**—A compound disk or compound cylinder containing a radial crack, subjected to point loads, is studied and the stress intensity factors are calculated. To approach the crack problem, the uncracked compound disk solution is sought first using the complex variable method. By superposition with the uncracked geometry solution and by integration of the corresponding dislocation solution, the crack problem can be formulated in terms of a singular integral equation, and the stress intensity factors of the radial crack can be evaluated accurately using a collocation technique. Extensive numerical results for both tensile and compressive point loads are obtained.

## INTRODUCTION

In this paper, a compound disk (or compound cylinder) containing a radial crack and subjected to point loads is studied, and the stress intensity factors are computed. Results reported herein, as a matter of fact, form part of the continuing work and development of the solution given in a previous work by the author (to appear a). In that work, a basic compound disk problem (a compound disk with an embedded edge dislocation) was solved and then the dislocation solution was used as Green's function to tackle general compound disk crack problems; as an example, stress intensity factors of a radial crack in a compound disk subjected to uniform crack surface pressure were calculated. In another paper by the author (to appear b), a rotating compound disk with a radial crack was considered. In this area, numerous related studies have been conducted. A brief discussion of some relevant work can be found in the previous paper, and some of the literature (Bowie and Freese, 1970; Bowie and Neal, 1970; Chang, 1983; Delale and Erdogan, 1982; Emery and Segedin, 1972; Isida, 1975; Labbens *et al.*, 1976; Erdogan and Gupta, 1975; Libatskii and Kovchyk, 1967; Rooke and Tweed, 1972; Tweed *et al.*, 1972; Tweed and Rooke, 1973; Tang and Erdogan, 1984; Tracy, 1975, 1979; Xu and Delale, to appear; Delale and Xu, to appear; Yarema, 1979) is documented in the references.

The geometry in the case under consideration is a compound disk consisting of an inner disk ($r \leqslant b$) and a bonded concentric annulus ($b \leqslant r \leqslant c$) with a radial crack embedded in the inner disk ($-b < a_1 \leqslant x \leqslant b_1 < b$, $\theta = 0$) (see Fig. 1). The materials of the inner

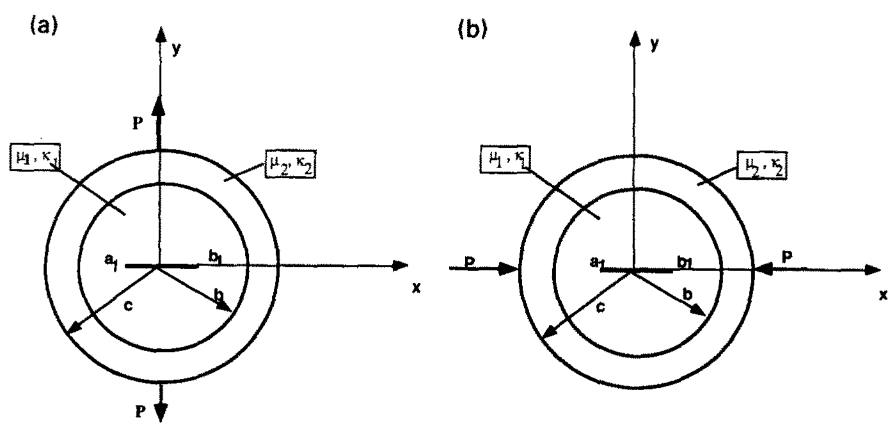


Fig. 1. A compound disk with a radial crack subjected to point loads.

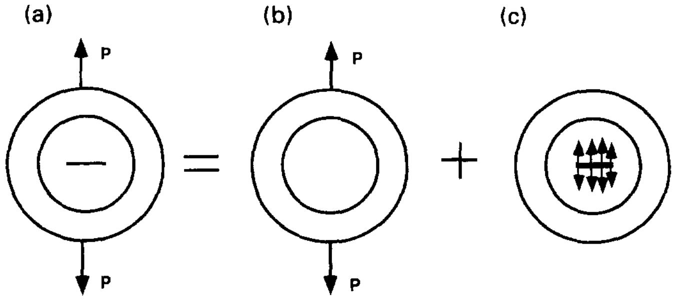


Fig. 2. Superposition of the crack problem.

disk and the outer annulus are assumed to be elastic and isotropic with mechanical constants $\mu_1$, $\kappa_1$ and $\mu_2$, $\kappa_2$, where $\mu_1$, $\mu_2$ are the shear moduli of the disk and the annulus respectively; $\kappa_i = 3-4\nu_i$ for plane strain, and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ $(i = 1,2)$ for the generalized plane stress case, and $\nu_i$ $(i = 1,2)$ is Poisson's ratio of the disk $(i = 1)$ or annulus $(i = 2)$. The loading is either a tensile point force $P$ applied at $\theta = \pm\pi/2$, $r = c$, or a compressive point force $-P$ applied at $\theta = 0$ and $\theta = \pi$, $r = c$.

To approach the crack problem, the elasticity solution of the uncracked compound disk subjected to point loads is sought using Muskhelishvili's complex variable technique and next, by superposition with the uncracked geometry solution, the crack problem under consideration may be reduced to a perturbation case in which the only loading is the crack surface pressure (see Fig. 2). Using the dislocation solution as a Green's function again, the stress intensity factors of the radial crack can be formulated in terms of a singular integral equation. Numerical results are obtained for both tensile and compressive loading cases. The effects of the disk–annulus relative stiffness and normalized annulus widths on the normalized stress intensity factors are also studied.

## FORMULATION OF UNCRACKED GEOMETRY CASE

In Muskhelishvili's complex variable technique, the plane problem is translated to seeking two complex potentials $\varphi(z)$ and $\psi(z)$ which satisfy all the boundary conditions. The stresses and displacements in polar coordinates can be expressed in terms of $\varphi(z)$ and $\psi(z)$ as follows:

$$\sigma_{rr}+\sigma_{\theta\theta} = 2[\varphi'(z)+\overline{\varphi'(z)}], \tag{1}$$

$$\sigma_{\theta\theta}-\sigma_{rr}+2\mathrm{i}\tau_{r\theta} = 2[\bar{z}\varphi''(z)+\psi'(z)]\,\mathrm{e}^{2i\theta}, \tag{2}$$

$$2\mu(u_r+\mathrm{i}u_\theta) = \mathrm{e}^{-i\theta}[\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}], \tag{3}$$

where $\mu$ is the shear modulus of the materials. $\kappa = 3-4\nu$ for plane strain; $\kappa = (3-\nu)/(1+\nu)$ for the generalized plane stress case, $\nu$ is Poisson's ratio of the material.

Subtracting (2) from (1), then taking the conjugate, we may obtain

$$\sigma_{rr}+\mathrm{i}\tau_{r\theta} = [\varphi'(z)+\overline{\varphi'(z)}]-[\overline{z\varphi''(z)}+\overline{\psi'(z)}]\,\mathrm{e}^{-2i\theta}. \tag{4}$$

On the boundary, we have the following relation:

$$\varphi'(t)+t\overline{\varphi'(t)}+\overline{\psi(t)} = \mathrm{i}(X+\mathrm{i}Y) = \mathrm{i}\int_{t_0}^{t}(X_n+\mathrm{i}Y_n)\,\mathrm{d}s, \tag{5}$$

where $t$ is the point on the boundary, $X$ and $Y$ are $x$ and $y$ components of the resultant force applied on the relevant boundary.

For the problem under discussion, we assume that $\varphi_1(z)$ and $\psi_1(z)$ are the stress functions for the inner disk and $\varphi_2(z)$ and $\psi_2(z)$ for the outer annulus. The boundary conditions may be written as follows:

$$\sigma_{rr1}+\mathrm{i}\tau_{r\theta 1} = \sigma_{rr2}+\mathrm{i}\tau_{r\theta 2}, \quad \text{on } r = b, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{6}$$

$$u_{r1}+\mathrm{i}u_{\theta 1} = u_{r2}+\mathrm{i}u_{\theta 2}, \quad \text{on } r = b, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{7}$$

$$\varphi_2'(t)+t\overline{\varphi_2'(t)}+\overline{\psi_2(t)} = \mathrm{i}(X+\mathrm{i}Y) = \mathrm{i}\int_{t_0}^{t}(X_n+\mathrm{i}Y_n)\,\mathrm{d}s, \quad t = c\,\mathrm{e}^{i\theta}, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{8}$$

where index 1 represents disk domain, index 2 represents annulus domain, and so on. We also have

$$F(\theta) = \mathrm{i}(X+\mathrm{i}Y) = \begin{cases} 0, & 0 \leqslant \theta < \dfrac{\pi}{2}, \\ -P, & \dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{3\pi}{2}, \\ 0, & \dfrac{3\pi}{2} < \theta \leqslant 2\pi, \end{cases} \quad \text{(tensile point loads).} \tag{9}$$

Since $\varphi_1(z)$ and $\psi_1(z)$ are holomorphic in the disk and on the circle $r = b$ as well, we can express them in terms of Taylor's series as follows

$$\varphi_1(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \psi_1(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{10}$$

For $\varphi_2(z)$ and $\psi_2(z)$, we can express them as

$$\varphi_2(z) = -\frac{X_b+\mathrm{i}Y_b}{2\pi(1+\kappa_2)}\log z+\sum_{n=-\infty}^{\infty} c_n z^n \quad \text{and} \quad \psi_2(z) = \frac{\kappa_2(X_b-\mathrm{i}Y_b)}{2\pi(1+\kappa_2)}\log z+\sum_{n=-\infty}^{\infty} d_n z^n, \tag{11}$$

where $X_b$ and $Y_b$ are $x$ and $y$ components of the resultant force applied on the circle $r = b$, and for the case under discussion, the resultant force must be zero. Therefore, we may write

$$\varphi_2(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{and} \quad \psi_2(z) = \sum_{n=-\infty}^{\infty} d_n z^n. \tag{12}$$

To proceed the solution by virtue of the Fourier series technique (Muskhelishvili, 1953), we expand eqn (9) as follows:

$$F(\theta) = \mathrm{i}(X+\mathrm{i}Y) = \sum_{n=-\infty}^{+\infty} A_n\,\mathrm{e}^{in\theta} \quad \text{with} \quad A_n = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)\,\mathrm{e}^{-in\theta}\,\mathrm{d}\theta. \tag{13}$$

It turns out that

$$A_n = \begin{cases} -\dfrac{P}{2}, & n = 0, \\ \dfrac{P}{\pi n}\sin\dfrac{n\pi}{2}, & n \neq 0. \end{cases} \tag{14}$$

Now, what remains is the determination of the coefficients in eqns (10) and (12).

Due to the symmetry of the geometry and the loads, the stresses and displacements must satisfy the following conditions:

$$
\begin{aligned}
(\sigma_{rri}+\mathrm{i}\tau_{r\theta i})_{\text{at point } z} &= (\sigma_{rri}-\mathrm{i}\tau_{r\theta i})_{\text{at point } \bar{z}}, \quad i=1,2,\\
(u_{ri}+\mathrm{i}u_{\theta i})_{\text{at point } z} &= (u_{ri}-\mathrm{i}u_{\theta i})_{\text{at point } \bar{z}}, \quad i=1,2,
\end{aligned} \tag{15}
$$

which can be satisfied only if all the coefficients in eqns (10) and (12) are real numbers.

Introducing the following transformation and applying the boundary conditions by substituting eqns (10), (12) into (6), (7) and (8). Let

$$
\alpha_n = a_n b^n, \quad \beta_n = b_n b^n, \quad (n=0,1,2,3,\ldots), \quad \gamma_n = c_n b^n, \quad \delta_n = d_n b^n, \quad (-\infty < n < +\infty),
$$

and define $\rho_1 = c/b$, thus we may obtain the following two sets of simultaneous equations:

$$
\begin{gathered}
2\alpha_1 - 2\gamma_1 - \delta_{-1} = 0, \quad \kappa_1\alpha_0 - 2\alpha_2 - \beta_0 - \frac{\mu_1}{\mu_2}\kappa_2\gamma_0 + 2\frac{\mu_1}{\mu_2}\gamma_2 + \frac{\mu_1}{\mu_2}\delta_0 = 0, \quad \alpha_2 - \gamma_2 - \delta_{-2} = 0,\\
\kappa_1\alpha_2 - \frac{\mu_1}{\mu_2}\kappa_2\gamma_2 + \frac{\mu_1}{\mu_2}\delta_{-2} = 0, \quad \gamma_0 + 2\rho_1^2\gamma_2 + \delta_0 = -\frac{P}{2}, \quad 2\rho_1\gamma_1 + \frac{\delta_{-1}}{\rho_1} = \frac{P}{\pi},\\
\rho_1^2\gamma_2 + \frac{\delta_{-2}}{\rho_1^2} = 0, \quad (\kappa_1 - 1)\alpha_1 - \frac{\mu_1}{\mu_2}(\kappa_2 - 1)\gamma_1 + \frac{\mu_1}{\mu_2}\delta_{-1} = 0,
\end{gathered} \tag{16}
$$

$$
\begin{gathered}
\alpha_n - \gamma_n + (n-2)\gamma_{-n+2} - \delta_{-n} = 0, \quad -n\alpha_n - \beta_{n-2} + n\gamma_n + \gamma_{-n+2} + \delta_{n-2} = 0,\\
\kappa_1\alpha_n - \frac{\mu_1}{\mu_2}\kappa_2\gamma_n - (n-2)\frac{\mu_1}{\mu_2}\gamma_{-n+2} + \frac{\mu_1}{\mu_2}\delta_{-n} = 0,\\
-n\alpha_n - \beta_{n-2} + \frac{\mu_1}{\mu_2}n\gamma_n - \frac{\mu_1}{\mu_2}\kappa_2\gamma_{-n+2} + \frac{\mu_1}{\mu_2}\delta_{n-2} = 0,\\
\rho_1^n\gamma_n - (n-2)\rho_1^{-n+2}\gamma_{-n+2} + \rho_1^{-n}\delta_{-n} = \frac{P\sin\dfrac{n\pi}{2}}{n\pi},\\
n\rho_1^n\gamma_n + \rho_1^{-n+2}\gamma_{-n+2} + \rho_1^{n-2}\delta_{n-2} = \frac{P\sin\dfrac{(n-2)\pi}{2}}{(n-2)\pi} \quad (n \geqslant 3).
\end{gathered} \tag{17}
$$

In eqn (16), there are 10 unknown constants but only eight equations. Due to the fact that the stress functions $\varphi_1(z)$ and $\varphi_2(z)$ can all differ from a complex constant without causing any change in the displacements, in other words, $\alpha_0$ and $\gamma_0$ are arbitrary constants, we can just let $\alpha_0 = \gamma_0 = 0$. By solving eqn (16) we obtain

$$
\begin{gathered}
\alpha_1 = \frac{P}{2\pi\rho_1} - \frac{P(\rho_1^2-1)[\mu_2(\kappa_1-1)+(1-\kappa_2)\mu_1]}{2\pi\rho_1[(\rho_1^2-1)\mu_2(\kappa_1-1)+\mu_1(2\rho_1^2-1+\kappa_2)]},\\
\alpha_2 = 0, \quad \beta_0 = -\frac{\mu_1 P}{\mu_2 2}, \quad \gamma_1 = \frac{P}{2\pi\rho_1} + \frac{P[\mu_2(\kappa_1-1)+(1-\kappa_2)\mu_1]}{2\pi\rho_1[(\rho_1^2-1)\mu_2(\kappa_1-1)+\mu_1(2\rho_1^2-1+\kappa_2)]},\\
\gamma_2 = 0, \quad \delta_{-2} = 0, \quad \delta_{-1} = -\frac{P\rho_1[\mu_2(\kappa_1-1)+(1-\kappa_2)\mu_1]}{\pi[(\rho_1^2-1)\mu_2(\kappa_1-1)+\mu_1(2\rho_1^2-1+\kappa_2)]}, \quad \delta_0 = -\frac{P}{2}.
\end{gathered} \tag{18}
$$

It may be seen in eqn (18) that $\alpha_2$, $\delta_{-2}$ and $\gamma_2$ are all zero. These results should be expected, and moreover, all the even indexical coefficients are expected to vanish from the following symmetry conditions:

$$\sigma_{rri}(r,\theta) = \sigma_{rri}(r,\pi-\theta), \quad \sigma_{\theta\theta i}(r,\theta) = \sigma_{\theta\theta i}(r,\pi-\theta), \quad \tau_{r\theta i}(r,\theta) = -\tau_{r\theta i}(r,\pi-\theta) \quad (i=1,2),$$
$$u_{ri}(r,\theta) = u_{ri}(r,\pi-\theta), \quad u_{\theta i}(r,\theta) = -u_{\theta i}(r,\pi-\theta) \quad (i=1,2),$$

or in the complex expressions,

$$(\sigma_{rri}+\mathrm{i}\tau_{r\theta i})_{\text{at point } z} = (\sigma_{rri}-\mathrm{i}\tau_{r\theta i})_{\text{at point } -\bar{z}}, \quad (i=1,2),$$
$$(u_{ri}+\mathrm{i}u_{\theta i})_{\text{at point } z} = (u_{ri}-\mathrm{i}u_{\theta i})_{\text{at point } -\bar{z}}, \quad (i=1,2). \tag{19}$$

After solving eqn (17), the stresses and displacements may be readily determined by eqns (1)–(4). The hoop stress $\sigma_{\theta\theta 1}(r,\theta)$ in the inner disk is found to be:

$$\sigma_{\theta\theta 1}(r,\theta) = \sum_1^\infty [n(n+1)a_n r^{n-1}\cos(n-1)\theta] + \sum_1^\infty [nb_n r^{n-1}\cos(n+1)\theta] \quad \text{(tensile loads)}. \tag{20a}$$

The hoop stress $\sigma_{\theta\theta 1(r,\theta)}$ for compressive loads can also be found as follows:

$$\sigma_{\theta\theta 1}(r,\theta) = \sum_1^\infty [2n(2n-1)(-1)^{n-1}a_{2n-1}r^{2n-2}\cos(n-1)\theta]$$
$$+\sum_1^\infty [(2n-1)(-1)^n b_{2n-1}r^{2n-2}\cos n\theta] \quad \text{(compressive loads)}. \tag{20b}$$

## CRACK PROBLEM

By superposition of the uncracked geometry solution obtained in the preceding section, we can reduce the original crack problem to a perturbation case. The boundary conditions of the perturbation problem for the case under consideration can be written as follows:

$$\sigma_{rr2}(c,\theta) = 0, \quad \tau_{r\theta 2}(c,\theta) = 0, \quad 0 < \theta \leqslant 2\pi, \tag{21}$$

$$\tau_{r\theta} = 0, \quad 0 \leqslant r \leqslant c, \quad \theta = 0 \quad \text{and} \quad \theta = \pi, \tag{22}$$

$$\sigma_{\theta\theta 1}(r,0) = p(r), \quad a_1 \leqslant r \leqslant b_1, \tag{23}$$

$$u_\theta(x,0) = 0, \quad -c \leqslant x \leqslant a_1 \quad \text{and} \quad b_1 \leqslant x \leqslant c, \tag{24}$$

where $p(r) = -\sigma_{\theta\theta 1}(r,0)$, and $\sigma_{\theta\theta 1}(r,0)$ is the disk hoop stress of the uncracked geometry solution and is given by eqn (20).

Defining

$$f(r) = \frac{\partial}{\partial r}[u_\theta(r,0^+) - u_\theta(r,0^-)], \quad (a_1 \leqslant r \leqslant b_1), \tag{25}$$

by integrating the dislocation solution and applying boundary condition (23), we may obtain the following singular integral equation:

$$\int_{a_1}^{b_1} \frac{f(t)}{t-x}\,\mathrm{d}t + \int_{a_1}^{b_1} \kappa(x,t) f(t)\,\mathrm{d}t = \frac{\pi(1+\kappa_1)}{2\mu_1} p(x), \quad a_1 \leqslant x \leqslant b_1, \tag{26}$$

with the Fredholm kernel $\kappa(x, t)$ given by

$$\kappa(x,t) = \frac{\pi(1+\kappa_1)}{2\mu_1}\left\{2c_{02}+6d_{12}x+\sum_{2}^{\infty}[n(n-1)x^{n-2}a_{n2}+(n+1)(n+2)x^n b_{n2}]\right\}, \quad (27)$$

where $c_{02}$, with unknowns $c_{03}$ and $b_{03}$, can be determined by the following simultaneous equations:

$$2c_{02}-\frac{b_{03}}{b^2}-2c_{03} = -\frac{2\mu_1 t}{\pi(1+\kappa_1)b^2},$$

$$\frac{(\kappa_1-1)b}{2\mu_1}c_{02}+\frac{1}{2\mu_2 b}b_{03}-\frac{(\kappa_2-1)b}{2\mu_2}c_{03} = \frac{t}{\pi(1+\kappa_1)b}, \quad \frac{b_{03}}{c^2}+2c_{03} = 0, \quad (28)$$

and $d_{12}$, with unknowns $c_{13}$, $d_{13}$ and $S_1$, may be determined by the following equations:

$$2bd_{12}+\frac{2c_{13}}{b^3}-2bd_{13} = -\frac{\mu_1}{\pi(1+\kappa_1)}\left(\frac{3t^2}{b^3}-\frac{2}{b}\right),$$

$$\frac{(\kappa_1-2)b^2}{2\mu_1}d_{12}-\frac{1}{2\mu_2 b^2}c_{13}-\frac{(\kappa_2-2)b^2}{2\mu_2}d_{13}+S_1 = \frac{1}{\pi(1+\kappa_1)}\left[\frac{3}{4}\frac{t^2}{b^2}-\frac{1+\kappa_1}{4}+\frac{\kappa_1-1}{2}\log_e b\right],$$

$$\frac{(\kappa_1+2)b^2}{2\mu_1}d_{12}-\frac{1}{2\mu_2 b^2}c_{13}-\frac{(\kappa_2+2)b^2}{2\mu_2}d_{13}-S_1 = \frac{1}{\pi(1+\kappa_1)}\left[\frac{3}{4}\frac{t^2}{b^2}-\frac{5+\kappa_1}{4}-\frac{\kappa_1-1}{2}\log_e b\right],$$

$$\frac{2c_{13}}{c^3}-2cd_{13} = 0, \quad (29)$$

and $a_{n2}$ and $b_{n2}$, together with unknowns $a_{n3}$, $b_{n3}$, $c_{n3}$ and $d_{n3}$ ($n \geqslant 2$), can be determined by:

$$n(1-n)b^{n-2}a_{n2}+(n+1)(2-n)b^n b_{n2}+n(n-1)b^{n-2}a_{n3}+(n+1)(n-2)b^n b_{n3}$$
$$+n(n+1)b^{-n-2}c_{n3}+(n-1)(n+2)b^{-n}d_{n3} = \frac{\mu_1 B_n}{\pi(1+\kappa_1)},$$

$$n(n-1)b^{n-2}a_{n2}+n(n+1)b^n b_{n2}-n(n-1)b^{n-2}a_{n3}-n(n+1)b^n b_{n3}$$
$$+n(n+1)b^{-n-2}c_{n3}+n(n-1)b^{-n}d_{n3} = -\frac{\mu_1 A_n}{\pi(1+\kappa_1)},$$

$$-\frac{1}{2\mu_1}nb^{n-1}a_{n2}+\frac{1}{2\mu_1}(\kappa_1-1-n)b^{n+1}b_{n2}+\frac{1}{2\mu_2}nb^{n-1}a_{n3}+\frac{1}{2\mu_2}(n-\kappa_2+1)b^{n+1}b_{n3}$$
$$-\frac{1}{2\mu_2}nb^{-n-1}c_{n3}-\frac{1}{2\mu_2}(n+\kappa_2-1)b^{-n+1}d_{n3} = \frac{C_n}{\pi(1+\kappa_1)},$$

$$\frac{1}{2\mu_1}nb^{n-1}a_{n2}+\frac{1}{2\mu_1}(n+1+\kappa_1)b^{n+1}b_{n2}-\frac{1}{2\mu_2}nb^{n-1}a_{n3}-\frac{1}{2\mu_2}(n+1+\kappa_2)b^{n+1}b_{n3}$$
$$-\frac{1}{2\mu_2}nb^{-n-1}c_{n3}-\frac{1}{2\mu_2}(n-1-\kappa_2)b^{-n+1}d_{n3} = \frac{D_n}{\pi(1+\kappa_1)},$$

$$n(n-1)c^{n-2}a_{n3}+(n+1)(n-2)c^n b_{n3}+n(n+1)c^{-n-2}c_{n3}+(n-1)(n+2)c^{-n}d_{n3} = 0,$$

$$n(n-1)c^{n-2}a_{n3}+n(n+1)c^n b_{n3}-n(n+1)c^{-n-2}c_{n3}-n(n-1)c^{-n}d_{n3} = 0 \quad (n \geqslant 2),$$

(30)

where $A_n$, $B_n$, $C_n$ and $D_n$ are given by

$$A_n(t) = \frac{(n+2)t^{n+1}}{b^{n+2}} - \frac{nt^{n-1}}{b^n}, \quad (n \geqslant 2), \tag{31}$$

$$B_n(t) = -\frac{(n+2)t^{n+1}}{b^{n+2}} + \frac{(n+2)t^{n-1}}{b^n}, \quad (n \geqslant 2), \tag{32}$$

$$C_n(t) = \frac{1}{2}\left[-\frac{(n+\kappa_1-1)t^{n-1}}{(n-1)b^{n-1}} + \frac{(n+2)t^{n+1}}{(n+1)b^{n+1}}\right] + \frac{[1+(-1)^{n+1}](1+\kappa_1)}{2(n^2-1)}, \quad (n \geqslant 2), \tag{33}$$

$$D_n(t) = \frac{1}{2}\left[-\frac{(n-1-\kappa_1)t^{n-1}}{(n-1)b^{n-1}} + \frac{(n+2)t^{n+1}}{(n+1)b^{n+1}}\right] - \frac{n(1+\kappa_1)[1+(-1)^{n+1}]}{2(n^2-1)}, \quad (n \geqslant 2). \tag{34}$$

From the definition of $f(t)$ in eqn (25), it follows that

$$\int_{a_1}^{b_1} f(t)\,\mathrm{d}t = 0. \tag{35}$$

The embedded crack has a square root singularity, and function $f(x)$ may be written as (Muskhelishvili, 1953; Erdogan, 1978):

$$f(x) = \frac{F(x)}{[(x-a_1)(b_1-x)]^{1/2}}, \tag{36}$$

where $F(x)$ is a bounded function at the crack tips. The stress intensity factors at crack tips $x = a_1$ and $x = b_1$ may be defined according to the conventional definition and calculated as follows

$$\begin{aligned} k_1(a_1) &= \lim_{x \to a_1^-} \sqrt{2(a_1-x)}\,\sigma_{\theta\theta 1}(x,0) \\ &= \frac{2\mu_1}{1+\kappa_1}\frac{F(a_1)}{\sqrt{(b_1-a_1)/2}}, \\ k_1(b_1) &= \lim_{x \to b_1^+} \sqrt{2(x-b_1)}\,\sigma_{\theta\theta 1}(x,0) \\ &= -\frac{2\mu_1}{1+\kappa_1}\frac{F(b_1)}{\sqrt{(b_1-a_1)/2}}. \end{aligned} \tag{37}$$

## RESULTS AND DISCUSSION

### *Special cases*

*Case 1.* If we let $c = b$, or let $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$, the problem under consideration reduces to a homogeneous disk embedded in a radial crack subjected to point forces. After some lengthy algebra, the closed form solutions of eqns (17) and (30) can be obtained. For example, in the case of $c = b$, the Fredholm kernel $k(r, t)$ in eqn (26) may be obtained:

$$k(r,t) = \frac{t}{b^2} + \frac{r-t}{b^2-tr} + \frac{t(t^2-b^2)+b^2r}{(b^2-tr)^2} + \frac{b^2t(t^2-2b^2)+b^4r}{(b^2-tr)^3}. \tag{38}$$

The uncracked geometry solutions [eqn (20)] are found to be

$$\sigma_{\theta\theta 1}(r,\theta) = \frac{2P}{\pi b}\,\frac{\left(1+\frac{2r^2}{b^2}\right)+\left(1+\frac{2r^2}{b^2}\right)^2\cos 2\theta+\frac{r^4}{b^4}\cos 4\theta}{\left(1+\frac{2r^2}{b^2}\cos 2\theta+\frac{r^4}{b^4}\right)^2}-\frac{P}{\pi b} \quad \text{(tensile loads)},$$

$$\sigma_{\theta\theta 1}(r,\theta) = -\frac{2P}{\pi b}\,\frac{\left(1+\frac{2r^2}{b^2}\right)-\left(1+\frac{2r^2}{b^2}\right)^2\cos 2\theta+\frac{r^4}{b^4}\cos 4\theta}{\left(1-\frac{2r^2}{b^2}\cos 2\theta+\frac{r^4}{b^4}\right)^2}+\frac{P}{\pi b} \quad \text{(compressive loads)}.$$

$$(39)$$

For the case of $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$, similar closed form expressions for $k(r, t)$ and $\sigma_{\theta\theta 1}$ can be obtained [replacing $b$ with $c$ in eqns (38) and (39)].

*Case 2.* If $c \to \infty$, the problem becomes a cracked disk or circular inclusion embedded in an infinite plane subjected to a pair of point forces at infinity. The closed form expression for kernel $k(r, t)$ in eqn (34) is found to be the same as Erdogan and Gupta's solution (1975) as follows:

$$k(r,t) = k_s(r,t)+k_f(r,t), \tag{40}$$

with

$$k_s(r,t) = \frac{A_3+A_4}{2}\,\frac{t}{rt-b^2}-\frac{A_3 r}{rt-b^2}+\frac{A_3(t^3-tb^2+b^2r)}{(rt-b^2)^2}+\frac{A_3(2b^4t-t^3b^2-b^4r)}{(rt-b^2)^3},$$

$$k_f(r,t) = -\frac{A_3A_6t}{b^2},$$

and

$$A_3 = \frac{\frac{\mu_1}{\mu_2}-1}{\frac{\mu_1}{\mu_2}+\kappa_1},\quad A_4 = \frac{\frac{\mu_1}{\mu_2}\kappa_2-\kappa_1}{\frac{\mu_1}{\mu_2}\kappa_1+1},\quad A_6 = \frac{2\left(1-\frac{\mu_1}{\mu_2}\right)}{\frac{2\mu_1}{\mu_2}+\kappa_1-1}.$$

Further, if $\mu_1 = \mu_2$, $\kappa_1 = \kappa_2$, the problem reduces to a homogeneous infinite plane with an embedded crack, the kernel $k(r, t)$ in eqn (40) becomes zero, leaving the singular integral equation (26) with just the Cauchy kernel term. Nevertheless, it can be readily shown that the stress component $\sigma_{\theta\theta 1}$ in eqn (20) will be zero when $c \to \infty$.

*Numerical results*

The singular integral equation (26) with the single-valuedness condition, eqn (35), can be solved using the Labatto–Chebyshev integration method developed by Erdogan and Gupta (1972) and Iaokimidis and Theocaris (1980), and the stress intensity factors can then be calculated from eqn (37). Extensive numerical results are obtained for both tensile and compressive loads. The effects of annulus widths $c/b$ and relative stiffness of the annulus $\mu_2/\mu_1$ on the normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ are investigated, and the results are presented in Figs 3–14. It may be seen that a stiff annulus will alleviate the inner radial crack propagation; when $c/b$ is increased from 1 to 4, the stress intensity factors decreases very rapidly. When $c = b$, or $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$, the results match Rooke and Tweed (1973a) and Delale and Xu (to appear) solutions almost exactly.

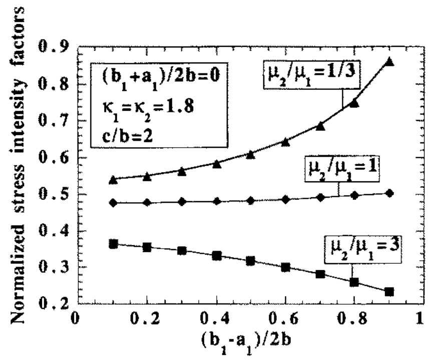


Fig. 3. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ vs normalized half crack lengths for various material stiffness combinations (tensile point loads, plane strain).

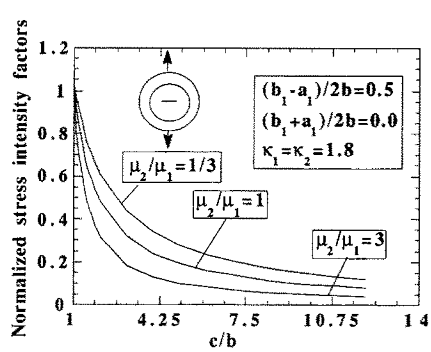


Fig. 4. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ vs normalized annulus width $c/b$ for various material stiffness combinations (plane strain).

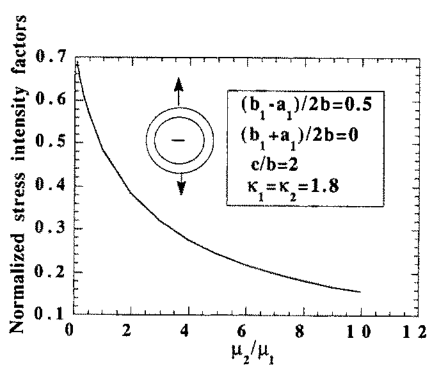


Fig. 5. Normalized stress intensity factors vs ratios of shear moduli $\mu_2/\mu_1$ (plane strain).

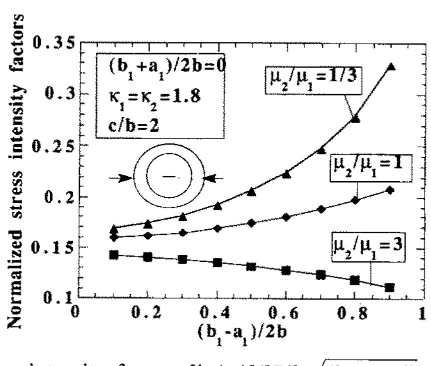


Fig. 6. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ vs normalized half crack lengths for various material stiffness combinations (plane strain).

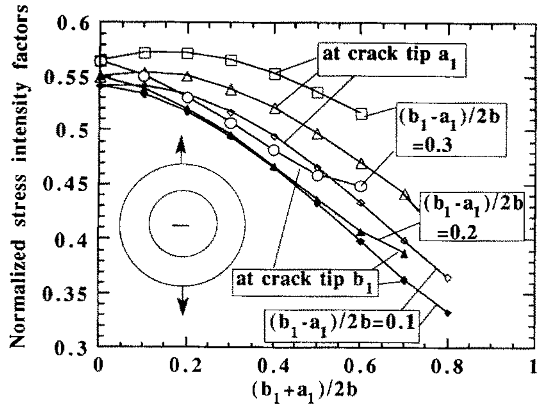


Fig. 7. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 1/3$, $\kappa_1 = \kappa_2 = 1.8$).

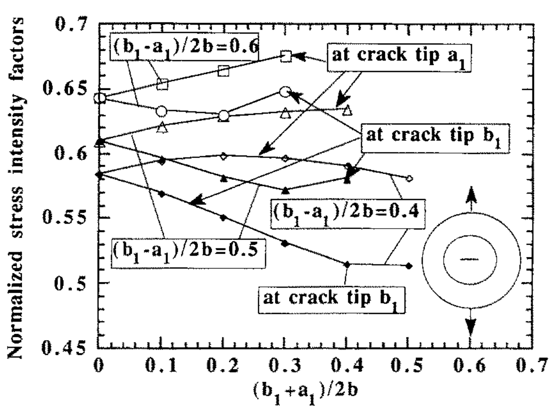


Fig. 8. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 1/3$, $\kappa_1 = \kappa_2 = 1.8$).

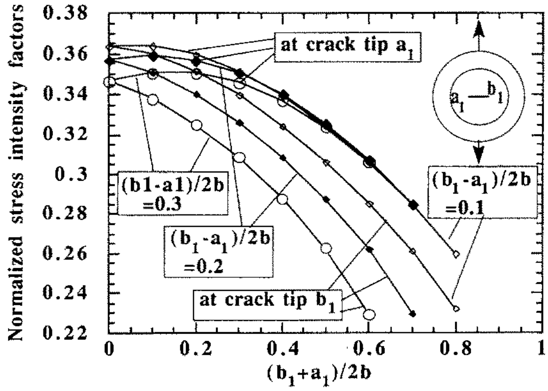


Fig. 9. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 3$, $\kappa_1 = \kappa_2 = 1.8$).

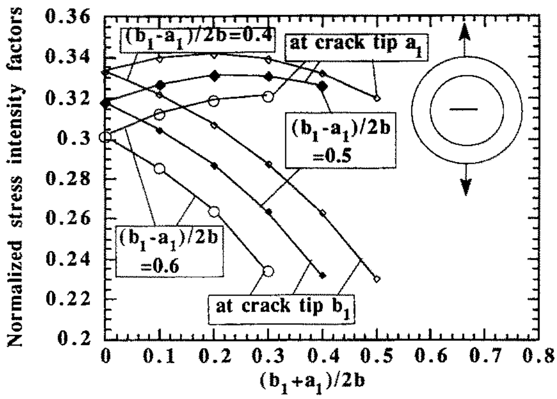


Fig. 10. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 3$, $\kappa_1 = \kappa_2 = 1.8$).

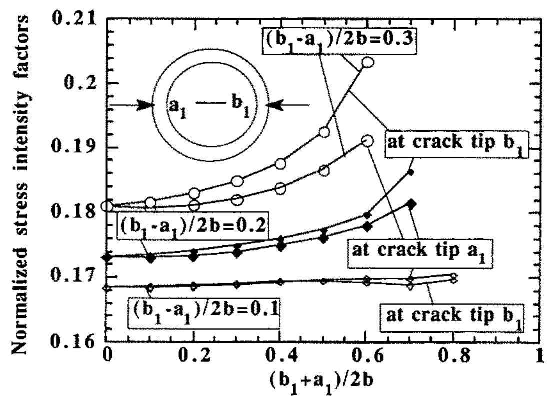


Fig. 11. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 1/3$, $\kappa_1 = \kappa_2 = 1.8$).

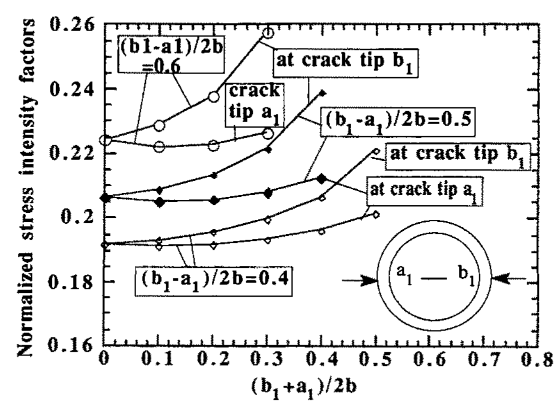


Fig. 12. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 1/3$, $\kappa_1 = \kappa_2 = 1.8$).

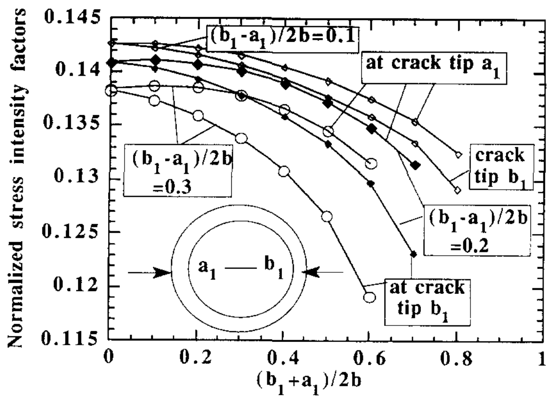


Fig. 13. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 3$, $\kappa_1 = \kappa_2 = 1.8$).

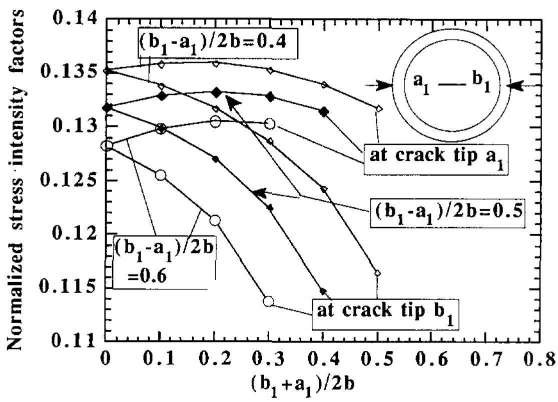


Fig. 14. Normalized stress intensity factors $[k_1(a_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ and $[k_1(b_1)]/[P/b\sqrt{(b_1-a_1)/2}]$ for various crack configurations ($c/b = 2$, plane strain, $\mu_2/\mu_1 = 3$, $\kappa_1 = \kappa_2 = 1.8$).

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